Teacher: What equation does this line represent?

If our students do not know the equation for the line, we tell them the letters used to represent all the ways there are to add two groups to ten.

Student: $x + y = 10$.
Teacher: What would the graph for $x + y = 9$ look like?
Student: It would be parallel to the one for ten, but it would cross the x and y axis at nine.

If the answer is not apparent to our students, we graph the ways to add two numbers to equal nine and then ask our students if they can now predict what the graph for $x + y = 8$ looks like.

Equations record patterns symbolically. Graphs record equations in picture form so that we may see the entire pattern—between and beyond the numbers that we use to deduce the formula. Graphing takes what is and shows what will be.

Graphing Lesson Four...

In Lesson Four, we used a function machine to introduce our students to independent and dependent variables. We used teacher-made and student-made stacks of cards to connect the numbers going in to the numbers coming out. We recorded the number pairs on data tables, and wrote formulas for the patterns in the cards. Patterns are formulas and formulas are algebra. Coordinate graphing is algebra, as well. For every patterned stack of cards there is a graph waiting to be made.

Teacher: You predicted the numbers out from the numbers we put into our function machine. Lets see if graphing numbers in and numbers out would make our predicting any easier.

(illustration 14-5-2)
(Tables from the last illustration in Lesson Four with a separate graph beneath each set of numbers.)

For each function graphed, we write the accompanying equation or formula.

Areas...

What else is graphable? In Lesson One, we asked: How many different sizes of squares and rectangles can we make using 24 S-1 squares, with no squares left over? We made a table of lengths and widths for all the areas of twenty-four that we found. We asked of the numbers in our table: Is there a pattern we can see in the lengths and widths that might tell us if we have found all the lengths and widths we can for an area of 24? We recorded the pattern for lengths and widths and areas as: $lw = a$.

We asked of our pattern: Could we use the formula to help us know when we have made all the different rectangles that can be made for a given number of S-1 squares? But we did not ask: If the formula $lw = a$ is true for rectangles that we make with S-1 squares, is it also true for the areas of rectangles that cannot be made with S-1 squares? We can graph the formula $lw = 24$ and see.

(illustration 14-5-3)
(Coordinate graph of the areas of 24. Length on y axis. Width on x axis. Plot the points for all the S-1 rectangles. Add to the graph the equation: $lw = 24$.)

A length of 9 and width of 2 2/3 will produce a rectangle with an area of 24. This rectangle is represented on our graph, but we cannot make it with our S-1 squares. Our graph of $lw = 24$ includes rectangles with areas of 24 that have fractions or decimals in their sides. Does our graph include all the lengths and widths for areas of 24?

Our graph also shows us that not all the points we plot produce straight lines.

The graph of $lw = 24$ is also the graph for $xy = 24$, the equation for all the ways there are to multiply any two numbers together to get 24. "Any two numbers" includes fractions and decimals. Algebra takes the pattern we find for rectangles and gives us a formula that connects to all of math.

$c = \pi d$...

In "The story told again..." (page 360) the teacher at the math conference gave the roving band of students an example of a use for algebra. Her example is appropriate for our class, as well.

The materials needed are circular objects and sheets of graph paper large enough to record diameters and circumferences. We ask our students to mark off diameters and circumferences on their sheets of graph paper and draw the line that the plotted points produce.

What can our students learn of circles everywhere from the objects that they graph? Algebra takes a pattern that we find for diameters and circumferences for objects in our class and connects the line we draw to every other circle that exists. Algebra is a very useful tool.

Number facts...

Any mathematical pattern we have seen before is graphable. The number facts our students learned for multiplication are ordered pairs. What would a graph of the times tables look like? What might the equation for each new set of numbers be?

(illustration 14-5-4)
(On the same coordinate grid, graphs for the twos, threes, fours, fives and sixes times tables. Label the lines with the algebraic equations x = 2y, x = 3y, x = 4y and so on.)

How does the graph for the multiplication number facts compare with the graph for addition number facts we did in "Ways to make ten..." above?

Toothpicks...

Our students used toothpicks throughout *Beginning Number* to create ways to make threes and fours and fives. Toothpicks can also be used to create growing patterns we can graph.

(illustration 14-5-5)
(Toothpick design: Step one: Simple four-sided box. Steps 2, 3, 4 and so on: Boxes, each of which shares a side with its neighbor from the previous step. For reference, write step 1, step 2, step 3, etc. beneath each new step.)

We can record the pattern's growth algebraically as x = 3y + 1. The number of toothpicks needed to make each new step can be predicted from a graph.

(illustration 14-5-6)
(Show the graph for the toothpick design with the formula written beneath the graph. The x axis represents the number of steps, the y axis represents toothpicks used.)

What might another toothpick design look like?

(illustration 14-5-7)
(Second toothpick design... first box has a roof. No other boxes have a roof added.)

We can graph the total toothpicks used to make each new step. The formula for the growth is x = 3y + 2.

What other toothpick patterns might we create, then graph, then record in formulas? Any kind we want.

(illustration 14-5-8)
(A variety of toothpick designs. Any kind of design at all, added to in a regular way. Include the formula for each design.)

Ways to make ten or nine or eight; patterned stacks of cards; charts of lengths, widths and areas; $c = \pi d$ and other formulas; number facts of all kinds; toothpick designs—algebra is the graphing of variables related in some patterned way. It does not matter which relationships our students graph, as long as they see the connection between the numbers and the line.

Lesson Six

Purpose	Learn to recognize the slope and intercept of an equation to be graphed.
Summary	Students graph equations in the y = mx or y = mx + b formats and look for slope and intercept patterns in the resulting lines.
Materials	Graph paper.
Topic	y = mx.
Topic	y = mx + b.

Connecting equations to the graphs...

We can see patterns and write equations for the patterns that we see. We can use numbers from the formulas to create lines for the patterns that the numbers represent. Patterns are everywhere around. We can also look at the graphs for the equations and the formulas to see the connections and the patterns they reveal.

Teacher: What would a graph of y = x look like?

(illustration 14-6-1)
(Coordinate graph of y = x.)

We graph y = x and talk about the resulting line. As we add in each new element, we observe what happens to the graphed line.

Teacher: What would a graph of y = 2x look like?

(illustration 14-6-2)
(Coordinate graph of y = 2x.)

Teacher: Look at the graphs for y = x and y = 2x and tell me what you think a graph for y = 3x might look like.

What other equation questions might we ask?

What patterns can we see in the graphs for y = x, y = 2x and y = 3x that we might use to predict the graphs for y = 4x or y = 5x before we draw them?
The general equation for these graphs is y = mx. What generalizations might we make about the effect of m?
What would a graph of y = x + 2 look like?
Can we predict the graph for y = x + 3 or y = x + 4 from the graph for y = x + 2?
The general equation for these graphs is y = x + b. What generalizations might we make about the effect of b?
What would a graph of y = 2x + 2 look like?
Can we predict the graph for y = 2x + 3 or y = 3x + 4?
The general equation for these graphs is y = mx + b. Do the generalizations we made earlier for m and b still hold true when m and b are both in the formula?
A graph's slope is the rate at which the line rises. What effect does m have upon the slope?
A graph's intercept is the point at which its line crosses the x or y axis. What does b tell us about the intercept?
What would happen to the slope or intercept when m or x or b or y are negative?
What patterns can we see?
How might a graphing calculator help us explore the equations above?

One pattern to be seen is that m is the slope and b is the y intercept. Once the pattern is seen, what use might it have? The pattern helps us know the formula that might have produced the graph.

(illustration 14-6-3)
(Show a data table. Along side, show the graph of the table's data. Indicate the slope and the intercept on the graph. Write the y = mx + b formula using the slope and intercept information from the graph.)

Patterns are everywhere around. Understanding slopes and intercepts help our students look at the graphs they create for growths or rates of change or other kinds of data to see the equations and the patterns that their graphs reveal.

Lesson Seven

Purpose	Learn to recognize the effect that exponents have on a graph.
Summary	Students graph equations in the $y = mx^2 + b$ or $y = mx^3 + b$ formats and look for patterns in the resulting lines.
Materials	Graph paper blackline—squares small enough to permit graphing of x^2 and x^3.
Topic	$y = mx^2 + b$.
Topic	$y = mx^3 + b$.

Exponents and roots—more equations to explore and graph...

In Lesson Six we explored the general equation for a line: $y = mx + b$. What might a graph of $y = mx^2 + b$ look like? Or $y = mx^3 + b$? In Lesson Seven, we learn about and then explore exponents as we continue studying equations and their graphs.

x^2 ...

Teacher: What is the fewest number of S-1 squares you need to make a square? This is not a trick question.
Student: One.
Teacher: What is the fewest number of S-1 squares you need to make the next largest Square? Show me with your squares.
Students: Four.
Teacher: The next largest? Show me with your squares.
Students: Nine.
Teacher: The next?

(illustration 14-7-1)
(Successively larger squares made from S-1 squares. The 1, 4, 9, 16, 25, 36, etc. pattern.)

Teacher: 1, 4, 9, 16, 25, and so on are called square numbers because they are the numbers it takes to make each next-size square. In mathematics, square numbers are written like this:

$$1^2 \quad 2^2 \quad 3^2 \quad 4^2 \quad 5^2$$

Teacher: We would read these numbers as one squared, two squared, three squared, and so on. Or, we could say one to the second power, two to the second power, and so on.
What does six squared (written as 6^2) mean?
Student: A square with six on each side.

If our students cannot provide the meaning, we provide it for them.

Teacher: What does six squared equal? Show me with your S-1 squares.
Students: Thirty-six.
Teacher: The square number pattern is recorded as:

$$y = x^2$$

What would a graph of $y = x^2$ look like?

(illustration 14-7-2)
(Graph of $y = x^2$.)

Teacher: What do the coordinates on our graph's curve represent?

Radicals...

To help our students speculate on the meaning of coordinates on the $y = x^2$ graph we show them how to find square roots. The square number pattern of 1, 4, 9, 16, 25, 36, 49, and so on, is easily seen with S-1 squares. The square root pattern is just as easily revealed. A square number is the area of a square. Square root is the name given to each square's side.

(illustration 14-7-3)
(Squares with the sides and areas labeled.)

To explore squares and square roots more easily, students need calculators with x^2 and $\sqrt{}$ keys.

Teacher: Enter the number 5 in your calculator and press the x^2 key. What number do you get?
Students: 25.
Teacher: Leave the 25 there and press the square root key.

The teacher says "square root" and draws the $\sqrt{}$ symbol on the overhead.

Teacher: What number do you get?

Students: 5.

Teacher: The x^2 key squares any number you put in. The square root key tells you the number that was squared. x^2 tells you the square's area. Square roots tell you the square's side. Any number with a square root sign in it is called a radical.

Numbers in the square number pattern of 1, 4, 9, 16, 25 and so on, are numbers that can be made with S-1 squares. The squares in the square number pattern are not the only squares that exist. The Power Block S-1, S-3 and S-5 squares can all be made with S-1 squares. The S-2 and S-4 squares cannot be made with S-1 squares, yet they are squares.

Teacher: With the help of the square root key on your calculator lets see if the coordinates on our graph's curve represent square numbers What is the area of the Power Block S-2 square?
Student: Two.
Teacher: The area is two. What is the length of each side? Use the square root key on your calculator to find out. You can round off the number to the first decimal place.
Students: 1.4
Teacher: Are the coordinates (1.4, 1.4) on our graph?
Students: Yes.
Teacher: What is the area of the Power Block S-4 shape?
Student: Eight.
Teacher: Use your calculator to find the length of each side.
Students: 2.8
Teacher: Are the coordinates (2.8, 2.8) on our graph?
Students: Yes.
Teacher: What other sides can we find for squares?

Familiar questions...

Additional questions that we ask in Lesson Seven follow the pattern we established in Lesson Six. We graph $y = x^2$ and talk about the resulting curve. As we add in each new element, we observe what happens to the curve.

Teacher: What would a graph of $y = 2x^2$ look like?

(illustration 14-7-4)
(Coordinate graph of $y = 2x^2$.)

Teacher: Look at the graphs for $y = x^2$ and $y = 2x^2$ and tell me what you think a graph for $y = 3x^2$ might look like.

Other familiar questions we might ask:

What patterns can we see in the graphs for $y = x^2$, $y = 2x^2$ and $y = 3x^2$ that we might use to predict the graphs for $y = 4x^2$ or $y = 5x^2$ before we draw them?
The general equation for these graphs is $y = mx^2$. What generalizations might we make about the effect of m on the graphs?
What would a graph of $y = x^2 + 2$ look like?
Can we predict the graph for $y = x^2 + 3$ or $y = x^2 + 4$ from the graph for $y = x^2 + 2$?
The general equation for these graphs is $y = x^2 + b$. What generalizations might we make about the effect of b on the graphs?
What would a graph of $y = 2x^2 + 2$ look like?
Can we predict the graph for $y = 2x^2 + 3$ or $y = 3x^2 + 4$ from the graph for $y = 2x^2 + 2$?
The general equation for these graphs is $y = mx^2 + b$. Do the generalizations we made earlier for m and b still hold true when m and b are both in the formula?
What would happen to the graph if m or x or b or y were negative?
What patterns can we see?

x^3...

Teacher: What is the fewest number of wooden cubes you need to make a cube? This is not a trick question.
Student: One.

Teacher: What is the fewest number of cubes you need to make the next largest cube? Show me with your wooden cubes.
Students: Eight.
Teacher: The next largest? Show me with your cubes.
Students: Twenty-seven.
Teacher: The next? You may need to share cubes with one another to build this cube.

(illustration 14-7-5)
(Successively larger cubes made from wooden cubes. The 1, 8, 27, 64. pattern.)

Teacher: 1, 8, 27, 64, and so on are called cube numbers because they are the numbers it takes to make each next-size cube. In mathematics, cube numbers are written like this:

$$1^3 \quad 2^3 \quad 3^3 \quad 4^3 \quad 5^3$$

We would read these numbers as one cubed, two cubed, three cubed, and so on. Or, we could say one to the third power, two to the third power, three to the third power and so on.
How many wooden cubes do you think it would take to make a cube with sides of five? What pattern do you see in the cubes that might help you to predict?

The pattern is side times side times side. If our students do not see this pattern, we guide their discovery.

Students: One hundred twenty-five.
Teacher: The cube number pattern is recorded as:

$$y = x^3$$

What would a graph of $y = x^3$ look like?

(illustration 14-7-6)
(Graph of $y = x^3$.)

Teacher: What do the coordinates on our graph's curve represent?

Familiar questions asked again...

The questions we asked in Lesson Six and for square numbers we ask again. We graph $y = x^3$ and talk about the resulting curve. As we add in each new element, we observe what happens to the curve.

Teacher: What would a graph of $y = 2x^3$ look like?
Look at the graphs for $y = x^3$ and $y = 2x^3$ and tell me what you think a graph for $y = 3x^3$ might look like.

Other familiar questions we might ask:

What patterns can we see in the graphs for $y = x^3$, $y = 2x^3$ and $y = 3x^3$ that we might use to predict the graphs for $y = 4x^3$ or $y = 5x^3$ before we draw them?
The general equation for these graphs is $y = mx^3$. What generalizations might we make about the effect of m on the graphs?
What would a graph of $y = x^3 + 2$ look like?
Can we predict the graph for $y = x^3 + 3$ or $y = x^3 + 4$ from the graph for $y = x^3 + 2$?
The general equation for these graphs is $y = x^3 + b$. What generalizations might we make about the effect of b on the graphs?
What would a graph of $y = 2x^3 + 2$ look like?
Can we predict the graph for $y = 2x^3 + 3$ or $y = 3x^3 + 4$ from the graph for $y = 2x^3 + 2$?
The general equation for these graphs is $y = mx^3 + b$. Do the generalizations we made earlier for m and b still hold true when m and b are both in the formula?
What would happen to the graph if m or x or b or y were negative?
What patterns emerge?

There are powers greater than 3, but we do not need to concern ourselves with all the possibilities. Students who wish to may explore $y = x^4$ on their own. x^4 cannot be modeled with squares or cubes, but abstractions are a part of algebra.

Graphing calculator...

If our resources include access to graphing calculators, students who wish to may experiment with a variety of equations and share their findings with the class. We have introduced exponents for x, but what happens when both x and y are squared? $x^2 + y^2 = b$ is the formula for a circle graph. What happens to the graph if the formula is $2x^2 + y^2 = b$ instead? What can our students discover with the aide of a graphing calculator or by plotting points on their own that we would not think to teach them? What might we learn be watching them explore?

Confusion...

One difficulty students encounter when exponents are introduced is confusing the meaning of numbers like 3x with x^3. As long as the numbers we present have meaning, confusions are easily undone.

Teacher: Please explain to me what 3x means.
Student: You have x three times.
Teacher: That's the same as x plus x plus x. Please explain to me what x^3 means.
Student: x time x times x.

If our students cannot answer these questions, we model what the numbers mean.

(illustration 14-7-7)
(A drawing of three identical squares placed side-by-side with x written inside each square and 3x written beneath the drawing. A drawing of a cube in 3D with x written on the three visible sides and x^3 written beneath the drawing.)

Can we model the difference between:

2x	**x^2**
2x + 2	**$2x^2$**
3x	**$3x^3$**
$2x^2$	**$(2x)^2$**
3x + 3	**$(x + 3)^3$**

The rule for parentheses is—do the operation inside the parenthesis first.

Lesson Eight

Purpose	Learn to multiply (x + y)(x + y) equations. Learn to connect the multiplying to concepts already understood.
Summary	We show our students how to use drawings to solve equations like (x + y)(x + y). We connect the algebra to arithmetic.
Materials	Graph paper.
Topic	(x + y)(x + y).
Topic	(x + y)(x + z).
Topic	(x + y)(z + w).
Topic	What comes next depends.

Solving equations—(x + y)(x + y)...

Algebra is the branch of mathematics in which letters represent basic arithmetic relationships. (x + y)(x + y), which can also be written as $(x + y)^2$, is an example of the kind of symbolic relationships that we associate with algebra. But often times, the equations are presented as if there were no connection to the arithmetic concepts our students already understand. In Lesson Eight, we explore the meaning of (x + y)(x + y) equations so that our students can understand the connection between the letters and the arithmetic the letters represent.

(10+2)(10+2)...

We can multiply 12 x 12 in a variety of ways. We can multiply 12 x 12 traditionally:

$$\begin{array}{r} 12 \\ \underline{\times 12} \\ 24 \\ \underline{120} \\ 144 \end{array}$$

Or, we can multiply each number separately and add:

12		
x12		
4	(2 x 2)	Bottom units times top units.
20	(2 x 10)	Bottom units times top tens.
20	(10 x 2)	Bottom tens times top units.
100	(10 x 10)	Bottom tens times top tens.
144		The numbers added

Or, we can write 12 x 12 as (10+2)(10+2) and multiply it the way we would multiply (x + y)(x + y).

We can illustrate 12 x 12 on a piece of graph paper:

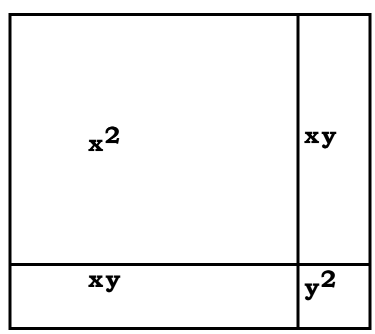
(illustration 14-8-1)
(Graph paper with a line drawn around a 12 by 12 square. Beneath the square write 12 x 12 = 144.)

We can also illustrate (10 + 2)(10 + 2) on a piece of graph paper:

(illustration 14-8-2)
(The same 12 by 12 square graph paper illustration from above with two additional lines drawn in. One line is drawn 10 squares over. The second line is drawn 10 squares down. The new lines form a 10 by 10 square, two 10 by 2 rectangles and a 2 by 2 square inside the 12 by 12 square. Beneath the illustration write: (10 + 2)(10 + 2) = 100 + 20 + 20 + 4.)

The problem can also illustrate algebraically in a more universal way:

(illustration 14-8-3)
((x + y)(x + y) illustrated. Label the outsides of the shape x and y as appropriate. Note: The actual book illustrations representing x and y multiplication will have x and y in correct proportions. All x sides of equal length and all y sides of equal length.)



$(x + y)(x + y) = x^2 + 2xy + y^2$

In algebra, letters written side by side mean multiply. What values might we substitute for x and y to see if our algebraic model for multiplying squares holds true? If x = 10 and y = 2, we can see that (x + y)(x + y) is equivalent to the (10 + 2)(10 + 2) example from above. $x^2 + 2xy + y^2$ is the same as $(10)^2 + 2(10x2) + (2)^2$, which is the same as 100 + 40 + 4, which is the same as 144. Does (x + y)(x + y) represent a multiplication pattern that holds true regardless of the values we substitute for x and y? Our students can test other values of x and y to see.

Multiplying...

The drawing shows our students that the answer to the problem (x + y)(x + y) is $x^2 + 2xy + y^2$. In the elementary grades, understanding is our goal. If we choose to, we may also link the understanding that the drawings represent to the abstract methods of finding answers prevalent in the higher grades.

The algebraic rules for multiplying (x + y)(x + y) are the same as the rules for multiplying 12 x 12. For 12 x 12 the multiplication looks like this:

$$\begin{array}{r} 12 \\ \underline{12} \\ 24 \\ \underline{12} \\ 144 \end{array}$$

For (x + y)(x + y) the multiplication looks like this:

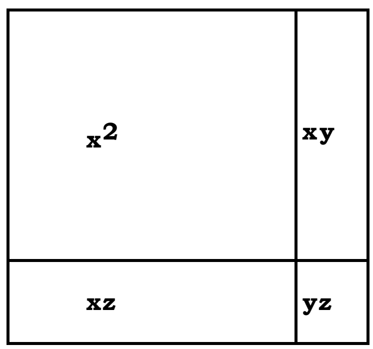
$$\begin{array}{r} x + y \\ \underline{x + y} \\ xy \;\; yy \\ \underline{xx \;\; xy } \\ xx \;\; 2xy \;\; yy \\ \text{or, } x^2 + 2xy + y^2 \end{array}$$

Algebra is present in the most basic problems that we give our students. We can choose to leave the algebra hidden or we can choose to bring it out.

Not square...

Can algebra present a model of multiplication of numbers that are not square?

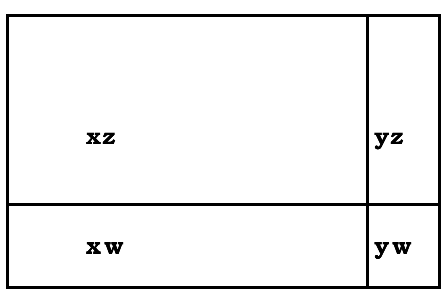
(illustration 14-8-4)
((x + y)(x + z) illustrated. Label the outsides of the shape x, y and x, z as appropriate.)



$(x + y)(x + z) = x^2 + xy + xz + yz$

Is the equation $(x + y)(x + z) = x^2 + xy + xz + yz$ true for all the possible values for x, y and z? Our students can substitute a variety of numerical values for x, y and z to see.

(illustration 14-8-5)
((x + y)(z + w) illustrated. Label the outsides of the shape x, y and z, w.)



$(x + y)(z + w) = xz + xw + yz + yw$

Is the equation $(x + y)(z + w) = xz + xw + yz + yw$ true for all the possible values for x, y, z and w? Our students can substitute a variety of numerical values for x, y, z and w to see.

Next...

What comes next depends upon what we think to ask and where the curiosity of our students leads. We may content ourselves with having given our students a background for understanding that $(x + y)(x + y) = x^2 + 2xy + y^2$. As readiness for algebra in the higher grades, this is a lot to know. If we have the time and the inclination though, we might give our students new questions to explore.

The questions we ask teach our students that there is a connection between the letters and the arithmetic the letters represent. When we use letters to record the patterns that we find, the word algebra describes the patterns that we see.

For every question we ask:

Do real numbers work in the algebraic answer that we find? Substitute any number for x to see.

Other questions we might ask:

The distributive principal for multiplication can be represented by the formula $a(b + c) = ab + ac$. Can we make a drawing that represents this formula? Of what use might this formula be?
The commutative principal for addition or multiplication can be represented as $(a + b) = (b + a)$ or $(a \times b) = (b \times a)$. Can we illustrate with manipulatives or with drawings what these two formulas represent? Of what use might these formulas be?
The associative principal for addition or multiplication can be represented as $(a + b) + c = a + (b + c)$ or $a \times (b \times c) = (a \times b) \times c$. The rule for parentheses is the operations inside the parentheses are done first. Can we illustrate with manipulatives or with drawings what these two formulas represent? Of what use might these formulas be?
Does $(y^2 + x^2)$ ever equal $(y + x)^2$?
$x + 3 = y$ is an equality. $x + 3 > y$ is an inequality. All the formulas we have illustrated so far have represented equalities. Can we create a drawing that represents an inequality?

Lesson Nine

Purpose	Learn that the ratios, rates, proportions and equivalencies from the fractions chapter are functional relationships.
Summary	Our students graph data they developed in their fraction lessons and use their graphs to create algebraic formulas.
Materials	Tables of data from earlier mathematical experiences.
Topic	Ratios.
Topic	Rates.
Topic	Proportions.
Topic	Equivalencies.

Ratios, rates, proportionality and equivalencies...

As our students refined their fraction skills in Chapter 11, they learned about ratios, rates, proportionality and equivalencies. Ratios, rates, proportions and equivalencies all pair two numbers together in a sequence so that each first number determines exactly one second number. All are functional relationships. When symbols are used to represent functions, functions are a part of algebra. Ratios, rates, proportionality and equivalencies represented symbolically are a part of algebra.

Ratios...

Ratios express the relationship of two numbers to each other. In Chapter 11 our students used their calculators to determine ratios arithmetically (page 263). Ratios can also be graphed coordinately.

Pi is the ratio between the diameter and the circumference of any circle. In Lesson Five we asked our students to plot diameters and circumferences on their sheets of graph paper and draw the line that the plotted points produced. The link between any ratio and algebra is the ratio's graph.

The ratio of 3 to 1 is 3. The ratio of 30 to 10 is 3 as well. We graph ratios by plotting the two numbers that produce the ratio. The points we plot for this ratio are (1, 3) and (10, 30).

(illustration 14-9-1)
(Graph of the points (1, 3) and (10, 30) with a line passing through the two labeled points.)

As we learned in Lessons Five and Six, the algebraic expression of this line is $y = 3x$.

The ratios we explored in the fractions chapter can be extended into algebra by graphing the numbers that we found. When the numbers graphed produce a line or curve we recognize, we can use the skills we learned in Lessons Five and Six to write the algebraic formula for the graph.

Shadow ratios

In the fractions chapter (page 264) we asked our students to find the length of the shadow of a stack of cubes 20 high, then 30, then 40. We asked if there were a relationship between the height of the cube stack and the length of its shadow. We made a chart of recorded shadow lengths for different heights of cubes. We asked our students if there were a relationship they could see that they could use to predict shadow lengths for stacks as yet unmade.

The shadow ratios we explored were a step away from algebra. If we graph the data from the chart of height and length, we can deduce the algebraic formula for the ratio.

(illustration 14-9-2)
(Show the chart of recorded shadow lengths for different lengths of cube stacks from the fractions chapter. Next to the table put the coordinate graph the stacks and their respective lengths. Draw the line through the various coordinate points. Write the formula for the line beneath the graph.)

We create an equation from the numbers known and use the equation to solve for numbers not yet known. Could our students use the formula to find shadow lengths for stacks not yet made? Would the formula help our students find heights from shadows cast? Could they measure heights of trees or buildings using the formula?

Bouncing balls

Drop a ball from a height. How high does it bounce? Drop the ball from higher up. Does it bounce higher? Is there a ratio between height of drop and height of bounce that will let us predict a ball's bounce from any height? These are questions that we asked for bouncing balls in the fractions chapter (page 265). In algebra we create graphs and formulas for the bouncing balls.

(illustration 14-9-3)
(Table of ball drop heights and bounce heights. Graph with each drop-height and bounce-height point as coordinates. Formula for the graph's line written underneath.)

Rates...

All the rates we explored in the fractions chapter are a part of algebra. Each rate table that we made can be graphed coordinately.

Teacher: My car holds 20 gallons in its tank. It gets 26 miles per gallon. What does the rate table look like for how far my car will go on two, three, or four gallons?

Gallons	**1**	**2**	**3**	**4**	**5**	**...**
Miles	**26**	**52**	**78**	**104**	**130**	**...**

For Algebra, we convert the rate table to a graph and use the graph to deduce the formula for the rate.

(illustration 14-9-4)
(Graph of the rate table's data. Formula for the graph written underneath.)

Each rate table that we made earlier we can graph.

Proportionality...

If we start with a rectangle with sides of 1 and 3 and we double both sides, the rectangle and its double are said to be proportional. Proportional means having the same ratio. In this case the ratio of the sides of the two rectangles is 2 to 1. These proportional rectangles share diagonals in common.

(illustration 14-9-5)
(Series of rectangles of successively larger size, all drawn with their lower left-hand corner in common, all sharing the same diagonal extending from their common lower left corner through their upper right corners. The first rectangle has sides of 1 and 3.)

Questions we might ask:

Do all rectangles that are proportional have diagonals in common?
Is it possible for rectangles to share diagonals and not be proportional?

Do other shapes that grow proportionally share the same diagonals?
If the sides grow proportionally, do the areas grow proportionally, as well?
Can we find algebraic formula for the graph the diagonals represent?

Equivalencies...

1/2 = 2/4 = 4/8 = 8/16

Teacher: Look at the numerators and the denominators of all these fractions and see what patterns you can find. Share with the rest of us any patterns that you see.

In the fractions chapter our students searched for patterns and connections in equivalencies (page 253). The separation between studying fractions and studying algebra is only in the questions that we ask. Any equivalency our students explore is an equivalency they can graph. For any graph, there is a formula to find.

(illustration 14-9-6)
(The points (1, 2), (2, 4), (4, 8), (8, 16) plotted and the connecting line drawn. Beneath the graph write the equation represented by the line.)

Algebra is the language that mathematicians use to record patterns and connections. It is how we write the rules for the patterns that we see.

Lesson Ten

Purpose	Learn to use algebra. Learn to connect algebra in school to algebra in real life.
Summary	We create an algebra environment in our room by making algebra a tool for finding out. We use opportunities already present in our student's lives.
Materials	Materials in our room and/or materials we find outside of class.
Topic	Algebra is in the environment we create.
Topic	Algebra is in the questions that we ask.
Homework	The work that we send home depends upon the work we do in class.

(23•32) — (13•32)...

A mother is helping her son with his algebra homework.

Mother: Show me the next problem.
Son: Its this one.

(23•32) — (13•32) =

Mother: Well, that's easy enough. You can see the answer is 320. Just write down 10 times 32 equals 320.
Son: Our teacher wants us to show all our work. We have to multiply out the numbers inside the parentheses and then subtract the second answer from the first.
Mother: Why would he want you to do all that extra work? You've already done problems just like this. You can just see what the answer is.
Son: No we haven't and no I can't.
Mother: Well, can you do this problem?

23x — 13x =

Son: Sure, that's easy. The answer is 10x.
Mother: Well then?
Son: Well then what?
Mother: Well then, you can see the answer to your homework problem is 10 times 32. All you have to do is substitute 32 for x.
Son: No they aren't. The one with the x's is for algebra. It only works with letters. The one with the 32's is for multiplication. I'll have to multiply the numbers first before I can subtract.

This mother knew that x in an algebraic equation could represent any number, including 32. This mother's son could not see that x represented more than a letter in a letter problem to be solved. Our goal for our students is that they be more like the mother than the son.

Using opportunities...

From the start, our purpose in presenting algebra to our students has been to provide a basic understanding of the underlying concepts. Basic understanding means our students know that algebra is more than x's in a formula. The focus of Lesson Ten is on making use of the algebraic skills our students have developed in Lessons One through Nine.

When we presented measurement to our students, we created an environment that helped our students learn measuring skills important in their world. We created this environment by using the measuring opportunities already present in our students' lives. In creating this environment, we knew we did not have to know exactly what to teach, because there was no one thing our students had to know. As we connect algebra in school to algebra in life, the process is the same. We use the algebraic opportunities already present in our students' lives to teach them algebra has a use.

What follows are sample lists of algebraic opportunities we may use to make our point. The lists are divided into opportunities for formulas and opportunities for graphs. The division is arbitrary, since formulas may be graphed and graphs lead to formulas.

Opportunities are as limitless as our imaginations and the imaginations of the students in our class. We add to the lists any situation that we encounter or question that we ask that can be graphed coordinately or represented by a formula.

y = mx + b...

The opportunities below represent uses for linear equations like $y = mx + b$. Students who have experienced Lessons One through Nine are collectively capable of discovering that the reverse of $y = mx$ is $x = y/m$. For the situations we present, we ask our students to devise formulas that express the relationships algebraically. Examples of formulas found by students in the past are included with each situation described.

Converting temperatures from Fahrenheit to Celsius. $F = 9/5C + 32$
Converting temperatures from Celsius to Fahrenheit. $C = 5/9(F — 32)$
Converting monetary and measuring units of all kinds:
- Yen to dollars: ¥ = r$, where r equals the rate of exchange.
- Dollars to yen: $ = ¥/r.
- Pounds to dollars. £ = r$.
- Dollars to pounds: $ = £/r.
- Meters to inches: m = 39.37 in.
- Kilometers to miles: k = 0.6214 mi.

Any metric measure converted to a U.S. measurement: $y = mx$ or $x = y/m$.
Miles per hour—how far would we travel in how many minutes or hours? Distance equals speed times time: $d = st$.
Miles per gallon equals miles traveled divided by gallons used: mpg = mi/gal.
Gallons needed for the trip equals miles to be traveled divided by miles per gallon: gal = mi/mpg.
Cost of filling up the tank—cost equals price per gallon times gallons pumped: $c = pg$.
Rates for everything from rental cars to phone fees to the cost of a plumber coming to our home. The formula is the base fee for the phone or car or service person plus the rate per minute, mile, or hour, times the minutes, miles, or hours used equals the total cost: $y = mx + b$, where y is the total cost, b is the basic fee, m is the rate per minute, mile, or hour, and x is the number of minutes, miles, or hours used.
Wage received for work—wage equals hourly rate of pay times hours worked: $w = rh$.
Lunch money collected—total collected equals number of students times cost of lunch: $t = nl$.
Money collected for any event or spent on anything in quantity. Total cost equals number of items purchased times the number of items bought.

If we choose to, we may introduce more elaborate algebraic thinking by asking our students to create age problems for their classmates to solve. The starting pieces of information might be the combined age of two people and how their individual ages compare:

Aaron and Jacob's ages total 23—The combined ages of the two.
Aaron is twice as old as Jacob plus one year—How their ages compare.
$J + A = 25$—The formula for combined ages.
$A = 2J + 1$—The formula for how their ages relate.
$J + (2J + 1) = 25$—Substituting the formula for A for the letter A.

$3J + 1 = 25$—Adding Js.
$3J = 24$—Understanding that taking 1 away from each side is okay to do.
$J = 8$—Figuring that if 3 Js are 24, then a single J is 8.
$A = 2J + 1$, $A = 2•8 + 1$, $A = 17$—Substituting the value of J into the formula.
Aaron is 17, Jacob is 8—Knowing what the ages are.

If we wish to give our imaginations a boost, we can look in any algebra or science book for the word problems and the formulas contained within, to see which problems and/or formulas relate to events in our students' lives. We remember as we look that the problems in mathematics that our students contemplate should come from their own lives and not from the life of Mr. Smith (page 150). Our assessment for our students is how good they become at finding uses for the mathematics that we teach. Our assessment for ourselves is how good we become at finding problems for our students to solve that do not mention Mr. Smith.

Coordinate graphs...

The equation for how many miles we travel each hour when we travel at a constant speed is $d = st$. But what is the equation for speed that is accelerating or decelerating? Formulas for acceleration and deceleration have their base in the formula $y = mx^2 + b$. Our students looked at graphs for $y = x^2$ in Lesson Seven, but they may not yet be as able to see the formula represented by a $y = mx^2$ graph as they can for graphs of $y = mx$.

Coordinate graphs can be useful in helping our students see relationships, even when our students cannot always see the formulas the graphs might represent. Sometimes we create graphs from formulas. Sometimes we create formulas from graphs. Sometimes we create the graph and leave deciding what the formula might be for a later year in school.

The first set of situations listed below represents opportunities for our students to use linear graphs to answer questions visually. The second set of situations represents opportunities for our students to create coordinate graphs from data for which they may not yet possess the knowledge to create the formula that the graph's line or curve might represent.

The first set

Fees charged at an hourly rate as compared to fees charged for the day. Parking lots, for example, may charge by the hour for every hour parked or charge a flat fee for cars that park all day. The formula for the hourly rate is $f = rh$ (fee equals rate times hours). The formula for the daily fee is $f = rd$ (fee equals rate times days). If the x axis is used to indicate hours and days (24 hours in a day), both formulas can be plotted on the same graph. Do the graphed lines for the two formulas intersect? If they do, what is the meaning of the coordinate at the intersecting point? Do the two lines show when it is better to pay the hourly rate and when the all-day rate is the better buy?

What other examples can we or our students think of where different rates for the same service can be visually compared by plotting different graphs? Can we use a graph for comparative shopping to help us know which of two items or services or fees is the better buy? Competing phone companies say one service plan is better than another. Can competing plans be graphed and then compared? Do lines representing different rates for the same goods or services always intersect?

Distances traveled at given rates of speed if we walk, run, drive, or fly, or travel at the sound of speed or light can all be graphed. One axis indicates the distance traveled, the other axis indicates the time. If we plot more than one line on the same graph can we see visually the difference it makes if we walk, run, drive or fly? How long an axis might we need to plot rates for walking and flying on the same graph? What scale or scales might we use?

Recycling fees for soda cans. Could a graph tell us how many cans we would need to collect to reach a pre-established goal? One axis is for numbers of cans, the other axis is for recycling fees. Can we use a graph to create a formula to tell us what the number of cans would be for any amount we might wish to earn?

Scale on a map. If 1 centimeter = 5 miles, the proportion or scale can be plotted and miles per centimeter read from the graph. One axis plots the centimeters, the other axis plots the miles.

How does a mother or father cooking at home know how much of an ingredient to add to a recipe for a family of six when the recipe is meant for two? How do the cafeteria cooks know how much of each ingredient to add to a recipe for several hundred children when the recipe may be meant for six? The recipe's proportions can be graphed and the quantities for different servings can be determined by looking at the graph. One axis plots the portions, the other plots the people to be served.

Even a leaky faucet is a graphing opportunity. If we measure the amount of water that leaks each minute, then plot each minute's leak, can we see how long it will be before the leaks fills a bucket? We would need to know the amount of water the bucket holds. What would we need to know besides the amount of water leaking each minute to know how long before the leak would flood our room? One axis records the amount of water leaked, the other axis records the time.

If we tied a small basket to a rubber band, added units of weight to the basket one at a time, and graphed how much more the band stretched with each new weight, would our graph let us predict how far the next weight would stretch the band? One axis records the number of weights, the other records the band's stretch. What does predicting the stretch have to do with the way that weighing machines work?

What else might we graph? The more experience we have, the more experiences we find.

The second set

If the amount added to a savings account is kept constant every week or month, the cumulative total of the money in the account can be shown on a $y = mx$ graph. But savings accounts earn interest and the interest earned earns interest, too. When interest and interest on interest are added to the savings, the graph is no longer linear. With the help of a calculator's percent key and some collective reasoning, students can calculate the new amount of money in an account each week or month for a constant amount saved and interest earned. The calculated amounts can be graphed and predictions made of future savings, even if a formula for calculating compound interest cannot yet be deduced from the graph's curve. One axis records the weeks or months that pass, the other records the cumulative amount. If the savings rate is maintained and the interest keeps adding up, how old would the child be when there is $1,000 in his or her account? $10,000? $100,000? More?

If we graph the census totals for population in the United States since the census takers began their count, what might the graph look like? Could we use the graph to predict what the total might be for the next count? One axis would plot the ten year intervals of the census, the other would plot the population count. If we can graph population growth of the United States, can we graph the population growth for other countries, or the world?

If we graph the temperature outside taken every hour of the day, can we see a pattern for the day? Is there a hottest time each day? Is the hottest time the same, regardless of how hot or cold the day might be? One axis plots the temperature, the other plots hourly intervals of time. We can put the graphs of all the days in a week on the same paper for easier comparison.

How long does it take a cup of boiling water to cool back down to the temperature of the room? If we take the temperature of the water in the cup every five minutes, what would our graph look like? Or, are five minute intervals too short or too long to measure change? One axis records temperature, the other records intervals of time.

How have Olympic records changed throughout the years? How have the times for, say, the 5,000 meter run or the 100 meter dash changed from the beginning until now? One axis plots the time, the other plots the year. What other records might we graph historically? How have the times for the mile run changed over the years? What has been the winning time for the Indianapolis 500 or the Boston Marathon over the years? What other events that have a history we can trace might we plot coordinately?

What happens to the breaking distances for cars as speeds increase? Is the graph of stopping distances linear or curved? One axis plots the speed before the breaks are applied, the other plots the distance traveled before the car comes to a complete stop. Since the children in our class are not drivers yet, the data we graph will have to come from some other source, like the State Highway Patrol. Are the breaking distances for all cars and trucks the same?

Other questions we might ask for breaking distances of cars: How do the police use the lengths of skid marks at a crash to tell how fast a car was traveling? What is the crash formula used to measure speed, based on how much a car was crushed in the wreck? What do speed and weight have to do with stopping distances of cars?

What is the average height in inches in each grade of the boys and girls who attend our school? One axis plots the grade. The other plots two lines: one for average height of boys, the other average height of girls. Are the two lines the same?

Any set of data that matches one number with another can be graphed coordinately. What other situations might we graph?

Science opportunities...

Seeds & Plants

As our students grow plants from seeds (page 397) to learn what makes a difference in the growth of plants, growth can be recorded coordinately on graphs. Can a graph of how much a plant grows each day be used to predict how tall the plant will be tomorrow or next week? One axis plots the plant's height, the other plots the days. Can the growth of more than one plant be plotted on the same graph? Will the growth be linear, or will the graph's path be raggedy or curved? Do the same kinds of plants grown by different students have the same kinds of growth lines?

Moonshine

As our students plot the path of the moon across the evening sky (page 406) the drawing paper that they use could just as easily be a graph. One axis records the times the observations are made, the other plots the moon's height above the horizon. Times measured can be at constant intervals on the same night or at a fixed time on successive nights. Can direction and height be plotted on the same graph? Can a graph of the moon's path tell our students where the moon was between the times they made their observations? Can our students use their graphs to predict where the moon will be at a future point in time? Are graphs of the moon's path across the sky linear or curved?

Pendulums

As our students study pendulums a lesson note suggests our students use a graph to help them predict when a pendulum might stop its swing (page 410). The graph suggested is a coordinate one.

Shadows

As our students study shadows (page 416) there were questions that we asked that could be answered by a graph. We asked our students to find a shadow of some object outside our classroom, like a tree or the roof of a building, and to mark the end of the shadow every five minutes. The shadow marks were to track the shadow's path. The shadow marks upon the ground are a graph from which predictions can be made. The shadow marks could be recorded on graph paper just as well. One axis plots the time, the other plots the distance or direction of the shadow's move. What other shadow events might our students graph coordinately?

Friction, force and motion

As our students observe the rolling of small carts as they study friction, force and motion (page 418) we can have them record their observations on coordinate graphs. One axis plots the height of the board down which the cart rolls measured in inches or degrees, the other plots the time it takes the cart to leave the board or the distance the cart travels before it comes to rest. Can the graph be used to predict times and distances for heights not yet tried? Data for adding weight to the cart can be graphed. Can the graph be used to predict the effect of weight on speed?

The lists of algebraic opportunities above are expanded by our imaginations and the opportunities that we find. If negative numbers have a meaning in our data, we use the negative quadrants on our graph. A graph converting Fahrenheit to Celsius converts for positive and negative, as well. A graph recording temperature for days and nights might also find a use for negative numbers in the winter time. If graphing calculators or computers are available for our use, we use the tools at hand. $y = mx + b$ equations can be represented on calculators and computers just as easily as can 2 + 2.

A smorgasbord...

When we slide our tray along the counter of a cafeteria, we know that the person next to us may make an entirely different set of choices, but we will both end up with a meal.

The ten lessons in this chapter present of activities from which to choose. Our purpose is to give our students a feeling for and a comfort with algebra in a general way. We choose activities to present that flow most naturally from our teaching and from our students' lives. The activities are not meant to limit the choices that we make. The point of the lessons is the exploration, not the rules. Confidence and facility can be achieved in many different ways.

Questions from Teachers

1. At what grade level should we begin teaching algebra?

(illustration 14-Q-1)
(S-5 Power Block along side two T-5 Power Blocks in the shape of an S-5.)

T-5 + T-5 = S-5 whether T-5 = 1 or S-5 = 1 or S-1 = 1. Our students learn this relationship as they explore the blocks. They learn to call the T-5 pieces halves when we say the S-5 piece is the whole. They learn to write the pieces as equations when we say to record the letters on the blocks. The lesson name for halves and wholes is fractions. The lesson name for equations written in letter form is algebra. Regardless of the lesson name we choose, two T-5 triangles still fit together to make an S-5 square.

Algebra in the elementary grades is not a set of skills to teach, it is an attitude—for our students and for ourselves as well. Regardless of the grade level, we let the learning flow naturally from the materials we have at hand. At what grade level should we begin? At any grade level where we teach math.