Chapter 6

Geometry, Shapes, Relationships and Constructions

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Before We Begin

Geometry is...

Geometry is:
City skylines.
Country skylines.
Bridges of all kinds.
Prints on gift wrap.
The way suitcases fit in the car or furniture fits in a moving van.
Parked cars.
Shapes of dollar bills and coins.
Shapes of soccer fields and hopscotch squares.
The location of the sun and moon and stars.
Houses, cars, fences, roads.
Art works of all kinds.
Tiles tessellating on the kitchen floor.
Blue prints of buildings to be built.
Power Blocks, Pattern Blocks, Geoblocks, geoboards.
Fractals.
Chaos.
Looking at the moon and stars and wondering.
More than Euclid's parlor games.
Anything with a line.
Anything with an angle.
Anything with a shape.
Anything that takes up space.
Anything that fits together.
Anything that does not.
Images in our minds.
Mathematics, not arithmetic.

We cannot design a home to live in or a road to travel on without geometry. We cannot fly from here to there without geometry. We cannot mark out a playing field, store groceries on a shelf, place food in the refrigerator, judge distances, or represent the world in drawings or pictures without geometry. We cannot scribble out directions on a scrap of paper to show a friend how to reach our home. We cannot manipulate objects inside our heads We cannot do much without geometry.

Spatial sense...

Geometry is shape and space, not number, yet geometry is equally as important as number in understanding the concepts of mathematics. Geometry is what we use inside our minds to picture math. Without geometric images in our heads, we cannot visualize, we can only memorize.

(illustration 6-0-1)
(Photo of a house. Child's drawing of the same house.)

How does a child know that the walls and roof of a real house can be represented by the rectangle and the triangle she draws on paper with her crayon? What in the child's mind has transformed the house seen into the house drawn? Spatial sense and visual imagery.

Imagine what a square looks like when it is cut in half and then cut in half again. Picture a shape formed by drawing a line equal distance around a single point. Envision a row of squares added to a rectangle of squares. Spatial sense is what we use to form these images in our minds.

Our understanding of mathematical relationships involves our ability to construct an image of the relationship in our mind. The better we become at constructing images, either in our heads or on paper, the better we are able to solve problems. Spatial sense and visual imagery play a major role in mathematical reasoning. Images and concepts go hand in hand. Spatial sense is geometric reasoning.

Already included...

Geometry is already included in nearly all the mathematics that students in Math Their Way or Math a Way of Thinking classrooms do. Geometry is a part of every manipulative material we use in math. Geometry is how we visualize the solutions to problems that we solve.

We use geometry in our classrooms when we:

Free explore with materials.
Make shapes with areas of two on our geoboards.
Sort and classify by shape or position or direction.
Draw pictures to represent problems in addition and subtraction.
Create graphs to record the data that we gather.
Measure or estimate.
Use squares to represent multiplication facts.
Illustrate the value of a fraction.
Demonstrate the value of Pi in algebra.
Construct anything with Power Blocks or Pattern Blocks.
Comprehend the patterns and connections in math.

Experiences from home...

The experiences our students bring with them from home are the experiences we build upon in school. The children who bring with them the richest experiences are the children who are most likely to succeed. Children who have engaged in frequent conversations at home will speak more fluently than children from homes where conversation is scarce. Children who have been read to every day at home will be better readers than children who have never heard or seen a parent read. Children who come from backgrounds rich in geometric experiences are much better prepared to learn mathematical concepts in school than children with less experience.

As teachers, we compensate for the different backgrounds our students bring with them to school by filling in the background not all parents may have known to provide. Our goal is that every child learn. We do not let any child's missing experiences keep us from teaching every child in our care.

Our lessons in geometry give all our students the same rich background of experiences in math. We maximize the background to maximize the learning opportunity.

Lesson One

Purpose	Provide a background in geometry equally for boys and girls, rich and poor while exploring shapes in geometry.
Summary	Students build as our questions focus their discoveries.
Materials	Building materials off all kinds: Lego blocks, Tinker Toys, Geoblocks, Pattern Blocks, Power Blocks, straws, toothpicks and clay, kindergarten building blocks.
Topic	Today is building day, let's see what you can build.
Topic	Each material used for building is a topic.
Homework	We encourage parents to provide building materials at home for boys and girls alike.

Building...

The dictionary defines geometry as a branch of mathematics dealing with the relations, properties and measurements of solids, surfaces, lines, points and angles. What does this definition mean? It means that geometry is about form and shape. What we learn from building is a sense of form and shape.

To develop a sense of shape, children must experience shapes. How lines and curves and angles form together to make shapes. How shapes relate to one another. Which shapes grow into different shapes as pieces fit together. Which shapes can be used to make their own shape in a larger form. Which shapes cannot. Building means experiencing forms and shapes.

Building means learning words to accompany the shapes we make. Rectangle, triangle, circle, square. Cube, sphere, cylinder, pyramid. Longer, shorter. Big, bigger, biggest. Small, smaller, smallest. Length, width, height, depth, volume, area, capacity, perimeter, circumference. And on and on.

Building means observing, investigating, experimenting, exploring, and thinking logically. Building means geometry. Geometry means readiness for understanding math.

The background children bring with them to school for talking is forged from how they have been spoken to and listened to. The background children bring with them to school for reading comes from being read to and being given the opportunity to read. The background children bring with them to school for geometry develops from their experiences with building. Building with Lego blocks, Tinker Toys, Power Blocks, Pattern Blocks and Geoblocks. Building with cardboard boxes of all kinds, with toothpicks, or with plastic straws. Building with anything around. By requiring every boy and girl in class to build during math, we ensure that all our students—male and female, rich and poor—gain the geometric and mathematical background that building can provide.

We take away the math from girls...

Children who come to school with backgrounds poor in math do not necessarily come from the poorest homes. Often the disadvantage brought to school from home is the disadvantage of being born a girl.

Traditionally in our culture, the boys receive the building blocks, the girls receive the dolls. Toys for boys come with the math built in. Toys for girls come with the math taken out. Lego blocks and Tinker Toys come with pictures and blueprints of things to build. Barbie Dolls come with pre-made cars and houses and pre-sewn clothes. We take away the math from girls when toys come in a final form.

When Barbie learns to live in a Lego house and drive around in a car made from an old shoe box, girls will have the same background in math that we already give to boys. We can help parents equalize the backgrounds of boys and girls by helping parents choose toys for girls and boys with the mathematics still left in. Girls need the opportunity to build at home and school as much as do the boys.

We connect...

We do not need to plan a lesson on building. We need only to permit the building to happen. We provide the materials and then say:

Teacher: Today is building day. Let's see what you can build.

(illustration 6-1-1)
(Collage of building materials in use. Kindergarten building blocks, Lego blocks, Tinker Toys, Geoblocks, Pattern Blocks, Power Blocks, straws, toothpicks and clay.)

Teacher: The buildings that you are making are an important part of mathematics. Nothing is built anywhere in the world without the help of math. There would be no roads or bridges or airports or cars or television sets or anything at all that is built by people if there were no math. We may not always see the math, but the math is always there.

When we teach our students that the time we set aside for building is called "mathematics," we teach them that every time they build something anywhere they are doing math. We connect school mathematics to the building that our students do at home.

In later lessons on geometry, we connect folding paper, looking in the mirror, drawing and art, and everything we see around us in our environment to mathematics. Our goal is that every child be aware of the mathematics in using paper, or looking into a mirror, or drawing, or simply looking around while walking home from school. More that being aware, the child will know that he or she is doing mathematics all the time and doing mathematics well.

With what?...

Teachers are resourceful. We build with whatever we can gather for our room. If we can only commandeer a box of straws from the cafeteria at school, then straws and string are the materials our students use.

(illustration 6-1-2)
(Straw and string constructions.)

If toothpicks and clay are all we have available for our students, then clay-and-toothpick buildings fill our room.

(illustration 6-1-3)
(Clay and toothpick structures.)

If we have Lego blocks and Tinker Toys, but not enough for all, then we set up building stations and let our students learn about taking turns.

(illustration 6-1-4)
(Stations for building with Lego blocks, Tinker Toys, straws, toothpicks, Pattern Blocks, Power Blocks and Geoblocks.)

Building means our students build with whatever we or they can find.

We buy the materials we can afford—teachers always do. We ask parents for contributions of building toys no longer in use at home. We ask our principal to see what textbook monies we can use to buy materials more appropriate for learning than many books on math. We ask stores in the community for donations of lumber scraps, or cardboard shipping boxes, or whatever else we might use. What building apparatus might a local toy store donate to our class?

As our students build, they exercise their imaginations actively. We exercise our imaginations just as actively as we search for ways to bring materials to our room.

Making and building...

Because beginning geometry is not a series of lessons, but a series of visual and tactile experiences, we allow our students to experience the materials on their own. Geometry in the elementary grades is a skill of hands, eyes and mind. For free exploration, we said: "Let's see what you can make." For geometry we say: "Let's see what you can build."

Is there really any difference between free exploration and building in geometry? As our students free explore with Power Blocks and Pattern Blocks, will they create different kinds of structures if we ask them to "build' instead of "make"?

In free exploration and in geometry, we roam around the room observing each child's creativity. We share our observations with the class. As our students work, we ask them questions that expand the range of their building. There is no limit to the questions we may ask because there is no limit to the possibilities our students show us. In free exploration and geometry, our questions focus the learning already taking place.

With experience and with age...

Children pass through a progression of skills as they learn to speak. How quickly and how well each child passes through the progression is a function of experience and age. We do not need to control the progression. We do not even need to worry about what the steps in the progression are. We need only assist the child by talking to her or him and listen to what he or she has to say.

Neither do we need to control the progression of skills that our students gain from building. We do not even need to know every mathematical implication of the building taking place. Just as we do not have to be linguists to help a child learn to talk, we do not have to be mathematicians to let our students build.

Building skills improve with experience and with age. A child at six is better at building than the same child was at four. The child at nine or ten is more skillful than the child at five or six. We provide the materials and the opportunity. Our students provide the learning on their own.

Words...

Children building at home give names to what they construct: house, car, garage, train station, tower, fort, road, airport hanger, hideout, zoo. Names are as limitless as the imagination of a child.

Names are important to building at school, as well. Names for shapes:

cube	square	pyramid
rectangle	prism	cone
cylinder	triangle	sphere
circle	hemisphere	hexagon
polygon	quadrilateral	polyhedral
parallelogram	pentagon	diamond
rhombus	trapezoid	ellipse

Names for properties:

horizontal	vertical	parallel
perpendicular	angles of all kinds	intersection
diameter	diagonal	circumference
perimeter	radius	vertex

face	side	edge
curve	volume	area
line	point	relationship

Facility with language is as important in geometry as it is in all of math and all of life. We teach the names of the objects and ideas that our students use. If we use a prism, then we give the prism a name. Growth of geometric vocabulary flows naturally from exploration and experience.

We ask questions. We teach names. We help our students to verbalize what is visualized and to visualize what is verbalized.

Lesson Two

Purpose	Expand the exploration of shape.
Summary	Students explore the properties of shapes guided by the questions that we ask.
Materials	Power Blocks, Geoboards, recording paper.
Topic	Power Blocks, what shapes make other shapes?
Topic	Geoboards, make shapes with 3 sides. 4 sides. 5 sides. More sides.
Topic	Which shapes can be made with Power Blocks & duplicated on a geoboard? Which shapes cannot?
Homework	If geoboards can be sent home, explorations are continued there.

Jill...

Building gives children experience with shapes, but building experiences are generalized and concrete. Geoboards help our students analyze the properties of shapes more systematically. They help our students translate observations into patterns and formulas and connect the concrete to the abstract. Geoboards are step two, not step one. Ryan (page 058), Hayley (page 059), and Jill represent examples of what can happen if we skip a step.

Instructor: What is this shape?

(illustration 6-2-1)
(Square formed on a geoboard by connecting the mid points of each side.)

Jill: A diamond.
Instructor: Not a square?
Jill: No.
Instructor: Make me a square on your geoboard.

(illustration 6-2-2)
(Rubber band all the way around the outside of a geoboard.)

Instructor: (Rotates Jill's board so that her square is now a diamond) What shape is this?

(illustration 6-2-3)
(The geoboard from the illustration immediately above, with the geoboard turned to put the square in the same "diamond" position as in the first "Jill" illustration above)

Jill: A diamond.
Instructor: Not a square?
Jill: No.
Instructor: Are squares and diamonds the same thing?
Jill: No.
Instructor: Then tell me when this diamond turns into a square.

(illustration 6-2-4)
(A sequence of pictures showing the geoboard "diamond" being rotated into the "square" position.)

Ryan is a third-grade boy. Hayley is a third-grade girl. Are we surprised by what they know or have yet to learn? In what grade do we think Jill might be? Are we surprised to learn that Jill is a teacher and not a child?

Why does Jill have so much difficulty in seeing the square in the diamond? If we gave Jill a fork and slowly moved the fork around, when would Jill think the fork became a knife? Why does Jill think a square will cease to be a square as it is slowly turned about? We may assume that knives and forks were a part of Jill's growing-up experience. We may assume as well, that Jill's growing-up experiences did not include manipulating squares.

Power Blocks and geoboards...

A geoboard is one of several tools we have for building a background in geometry. Geoboards work best when also linked to Power Blocks. Geoboards are step two. Power Blocks are step one. To connect the two we ask:

Which Power Block shapes make other shapes?
What kinds of shapes do Power Blocks fit together to make?
What shapes can you make with your Power Blocks that you can also make on your geoboard? Which shapes can you not?
What shapes can you make on your geoboard that you can also make with your Power Blocks? Which shapes can you not?
Can you make a geoboard shape with three sides?
A shape with four?
A shape with five or six or seven?
How far can you go?
What do we mean by "side"?
On our geoboards, shall we define "side" for shapes as a line that turns at a nail or a line that crosses anywhere?

(illustration 6-2-5)
(Show sides defined as turning at a nail and sides defined by rubber bands that cross between nails.)

What shall we count as a shape? Are open and closed figures both called shapes, or are shapes only those figures that can hold water?

(illustration 6-2-6)
(Show what is meant by open and closed figures and what is meant by "holding water")

For every question that we ask, we can add:

Keep a written record of what you find.

Lesson Three

Purpose	Learn to recognize reflective symmetry in shapes.
Summary	Students explore lines of symmetry with materials and mirrors.
Materials	Mirrors, hinged and regular, Pattern Blocks, Power Blocks, paper for tracing shapes, colored construction paper, lined paper.
Topic	Free exploration with mirrors.
Topic	Pattern Blocks and mirrors - exploring symmetry.
Topic	Power Blocks and mirrors - exploring symmetry.
Topic	Lines of symmetry in the room - make a list.
Topic	Free exploration with hinged mirrors.
Topic	Symmetry with Pattern Blocks and hinged mirrors.
Topic	Symmetry with Power Blocks and hinged mirrors.
Topic	Kaleidoscope - three mirror exploration
Homework	The search for symmetry is sent home.

Symmetry...

Symmetry is a geometric form that patterns take.

Simple definition of symmetry: Balanced proportions: beauty of form arising from such harmony.
Webster's New Collegiate Dictionary

More complex definition: A plane α is a plane of symmetry for a polyhedral P if and only if $M_{\alpha}(P)=P$. A line l is an axis of symmetry of a polyhedral P if the rotation R_l about axis l maps P onto itself. That is, $R_l(P)=P$.

Geometry: An Investigative Approach

The study of symmetry can be simple or complex. When we start with the simple, symmetry need not be complex.

Free exploring...

Teacher: Tell me what you can find out about your mirrors.

(illustration 6-3-1)
(Collage of free exploration activities with children and mirrors.)

Teacher: Can you see someone else's eyes in your mirror without that person's seeing yours?
Is what you see in your mirror on the surface of the mirror or inside? How do you know?
Can you make a periscope?

Pattern Blocks...

Teacher: Make a design using five Pattern Blocks. Put your mirror along side your design. Then, on the other side of your mirror, build the design that you see in the mirror. Lift up your mirror to see if the shape you built is exactly the same as the design you saw inside the mirror. If it is, congratulate yourself. If it is not, then try again to build exactly what you see.

(illustration 6-3-2)
(Show it done correctly. Start with a mirror with five Pattern Blocks in front. Step two: five pattern blocks on the other side of the mirror reproducing the design. Step three: The Pattern Blocks with the mirror removed.)

Teacher: The design you made is called "symmetrical." This kind of symmetrical means each half of the design is the mirror image of the other half. The mirror line down the middle is called the "line of symmetry" or the "axis of symmetry."
See if you can make a design that has a line of symmetry. That means when you put your mirror in the middle of your design, the half you see in the mirror is the same as the other half of your design.

(illustration 6-3-3)
(Collage of small Pattern Block shapes with lines of symmetry. Show one or two of the examples being checked for symmetry with a mirror.)

Teacher: How many different symmetrical designs can you make?
Can you make one with an axis of symmetry longer than your mirror?
Which of your Pattern Block pieces have lines of symmetry? Trace your pieces and record the lines of symmetry that you find.

(illustration 6-3-4)
(Pattern Block pieces traced with multiple lines of symmetry drawn.)

Teacher: Which of the Pattern Block pieces have the most lines of symmetry?
Are there any pieces with no lines of symmetry at all?
Can you make a design that has more than one line of symmetry?
Make a design that has two lines of symmetry. Then see if you can make one that has three. Then one with four, then one with five and so on.

(illustration 6-3-5)
(Different designs representing two, then three, then four, then five, then six lines of symmetry.)

Power Blocks...

Teacher: What lines of symmetry can you find in Power Blocks?

(illustration 6-3-6)
(Basic Power Block shapes with lines of symmetry drawn in or not drawn in as appropriate.)

Teacher: Does the size of the shape have any effect on the lines of symmetry that you can find?

Can you make symmetrical designs with Power Blocks, even when the pieces you might use have no lines of symmetry by themselves?

(illustration 6-3-7)
(Symmetrical Power Block design using non symmetrical pieces.)

Wherever...

Teacher: Where in our classroom can you find lines of symmetry?
Work with a partner and make a list of all the examples of symmetry that you find.
What lines of symmetry can you find outside of school today, wherever you might go?

Hinged mirrors...

(illustration 6-3-8)
(Two mirrors taped (or hinged) together.)

Teacher: Tell me what you can find out about your hinged mirrors.

(illustration 6-3-9)
(Collage of free exploration activities with children and hinged mirrors.)

Teacher: What happens as you move your mirrors together or apart?
What happens to objects you place between the mirrors as you move the mirrors?
What is the most reflections you can find for an object between your hinged mirrors?
Put one Pattern Block inside your hinged mirrors. Adjust your mirrors so that you see only the reflections of the whole piece inside, and not just parts of the piece. Now, around your mirror, build the design you see inside. When you have finished building, look in your mirror again, then lift up your hinged mirror slowly and see if the shape you have built is exactly the same as the design you see inside.

(illustration 6-3-10)
(Show a hinged mirror with one Pattern Block inside. Show the design built around the outside of the mirrors. Show the design with the mirrors removed.)

Teacher: How many different designs can you make that look like the designs you see inside your hinged mirrors? You may use more than one piece.

(illustration 6-3-11)
(Collage of Pattern Block hinged mirror designs. Show the piece between the mirrors and the angle of the mirrors. Show the design constructed next to the mirrors. Give examples of the effect on the designs of different mirror angles.)

Teacher: Do your designs have lines of symmetry?
What happens to the number of lines of symmetry as you move your mirrors?
Can you put your mirror on part of the shape so that all of the shape appears in your mirror?
What happens when you use Power Blocks?

(illustration 6-3-12)
(Collage of Power Block designs made to match what is seen within the hinged mirrors.)

If I asked you the same questions for Power Blocks as I did for Pattern Blocks, would your answers be the same as well?

Three-mirror kaleidoscope...

(illustration 6-3-13)
(Three mirrors taped (or hinged) together. Show the three taped mirrors in a line and also in the shape of an equilateral triangular prism. The caption indicates that the mirrors are taped together, but not in the closed position, so the size of the triangular prism may be changed at will. Demonstrate a change in size of the triangular prism.)

Teacher: I have given each of you a piece of lined paper and a piece of colored construction paper. Put your three-mirror kaleidoscope on top of one of the pieces of paper or on the two pieces side by side and see what you can see. It is okay to move the mirrors around on the papers. It is also okay to change the triangular shape the mirrors make.

(illustration 6-3-14)

(Drawings of the kinds of tessellating shapes and designs the students can create by manipulating their mirrors on the papers. The drawings will have to be computer generated to correctly simulate the designs inside the three hinged mirrors.)

Teacher: What else can you find out about your mirrors?

Experiences with symmetry...

Experience with physical objects lays the foundation for later generalization and abstraction of geometric ideas. We can know something intuitively long before we can prove what we know.

Experience with symmetry provides our students with geometric-pattern searching tools. The mirrors of Lesson Three teach our students reflective symmetry. The tessellations of Lesson Four teach our students that there are more kinds of symmetry than one.

Lesson Four

Purpose	Learn which polygons tessellate the plane, while discovering more kinds of symmetry.
Summary	Students explore shapes that tessellate and shapes that do not. They create their own tessellating shapes and turn them into Escher-like designs.
Materials	Power Blocks, Pattern Blocks, tag board cutouts, or templates. Blackline for tessellating shapes with angles written in and the shapes labeled.
Topic	Which polygons tessellate the plane.
Topic	Patterns for the polygons that tessellate.
Topic	Cutting tessellating polygons Escher style.
Topic	The evolution is from math to art.
Homework	Escher cutouts made at school go home for further exploration.

Tessellate...

A plane is a flat surface that may extend forever in all directions. A polygon is a closed figure with straight sides. A tessellation is a repeating pattern on a plane with no gaps between congruent figures and no overlapping. A tessellation of polygons is a repeating polygon pattern on a plane without gaps or overlaps.

Tessellating patterns are a part of nature.

(illustration 6-4-1)
(Collage of tessellating patterns in nature. Examples: Pineapple or pine cone surface, honeycomb, spider web, sunflower, cross section of a Nautilus shell, dry creek bed, turtle shell designs, etc.)

Tessellating patterns are a part of the structures humans make.

(illustration 6-4-2)
(Collage of human-made tessellating patterns. Illustrate a variety of geometric shapes. Examples: Floor tiles, windows on a skyscraper, Epcot Center dome, triangular surface of the San Jose Arena, Escher-like design, etc.)

Patterns in mathematics exist whether we assign numbers to them or not.

Questions for our students to explore:

Teacher: Which Pattern Block shapes tessellate the plane? Which Pattern Block pieces fit together with no holes in-between and no pieces on top of any other? I know the orange squares do.

(illustration 6-4-3)
(Tessellating Pattern Block squares.)

Teacher: Which other pieces do?
Student: Do we have to use just one kind of piece?
Teacher: For now, limit yourself to using the same pieces for each tessellation you make.

(illustration 6-4-4)
(Pattern Block tessellations with only the same kinds of pieces used to form regular tessellations.)

Teacher: Which Power Block shapes tessellate the plane? Which Power Block pieces fit together with no holes in-between and no pieces on top of any other? I know the T-2 triangles do.

(illustration 6-4-5)
(Tessellating Power Block multi-colored T-2 triangles.)

Teacher: Which other pieces do?
Student: Do we still have to use just one kind of piece?
Teacher: Yes, for now the limit is still just one kind of piece for each different tessellation that you make.

(illustration 6-4-6)
(Multi-colored Power Block tessellations with only the same kinds of pieces used to form regular tessellations.)

Teacher: Which other polygons tessellate the plane? Which geometric shapes can you think of that are not in your Pattern Blocks or Power Blocks that might tessellate the plane?

Do all polygons tessellate the plane? Here are polygons we might have our students try:

(illustration 6-4-7)
(Blackline of the following kinds of shapes. Shapes named in the blackline.)

rectangle	parallelogram	trapezoid
kite	4 or 5 quadrilaterals	equilateral triangle
isosceles triangle	scalene triangle	right triangle
pentagon (5)	hexagon (6)	heptagon (7)
octagon (8)	nonagon (9)	decagon (10)
dodecagon (12)	boomerang shape	

Patterns for the polygons...

Teacher: Can we find a pattern for the polygons that tessellate and the polygons that do not?

There is a pattern to be found. The basis of the pattern is the degrees in the angles of the tessellating shapes. Shapes that tessellate have angles or corners that fit together with no gaps or overlaps around a single point. Since there are 360° around a point, shapes that, when placed side-by-side, have angles which add to 360° are shapes that tessellate.

(illustration 6-4-8)
(Show that there are 360° around a point. Show the angles of tessellating square, triangle (two or three different ones), rectangle, parallelogram and trapezoid adding to 360° around a point.)

Quadrilaterals—four sided shapes—are shapes that tessellate the plane. Angles of a quadrilateral flipped or rotated around a point will add to 360°. The angles of a triangle add to 180°. Any two triangles of the same size and shape form a quadrilateral. Triangles and quadrilaterals are shapes that tessellate the plane. Any triangle. Any quadrilateral.

(illustration 6-4-9)
(Show examples of tessellating quadrilaterals. Show what is meant by any two triangles of the same size or shape forming a quadrilateral, Include the angles of the shapes.)

We do not explore angles or degrees until Lesson Seven, but the blackline of shapes has the angles written in. Degrees are the basis of the pattern, but shapes fit together or do not whether we can measure angles or not. We ask our students what patterns they can find in the sums of all the angles around a tessellating point. Can our students see a pattern in the angles or degrees? Will the pattern they might see help them know which other shapes might tessellate and which might not?

Four types...

Squares tessellate the plane as we lay the squares side by side. Other shapes may need to be flipped, rotated, or slid left or right before they tessellate. When our students have the opportunity to free explore and build with shapes, they learn that shapes can fit together in many ways. They may not know the techniques that they use are called flipping or rotating or sliding. If we wish to teach the words, we may.

Four types of symmetry form the patterns of geometry:

Reflectional
Rotational
Translational
Glide-reflectional

Reflectional: shapes held up against a mirror.

(illustration 6-4-10)
(Show a mirror example. Show with Power Blocks or Pattern Blocks.)

Rotational: mirrors that are hinged or shapes that are traced and turned.

(illustration 6-4-11)
(Show a hinged mirror example and a transparency example. Show with Power Blocks or Pattern Blocks.)

Translational: slides or glides, no turns, no flips, no mirrors used.

(illustration 6-4-12)
(Show an example. Show with Power Blocks or Pattern Blocks.)

Glide-reflectional: slides and flips together like foot prints in the sand.

(illustration 6-4-13)
(Show a foot print example. Indicate in the caption that glide-reflection is different than glides and reflections separately. Show with Power Blocks or Pattern Blocks.)

Visual explorations...

Teacher: What happens if we cut a piece from the right side of a square and put it on the left? Will the new shape still tessellate?

(illustration 6-4-14)
(Cut a triangle piece from the right side of a square and put the piece on the left side. Then show the shape tessellating.)

Teacher: Will a curved piece cut and moved still tessellate?

(illustration 6-4-15)
(Cut a wavy curved piece from the right side of a square and put the piece on the left side. Then show the shape tessellating.)

Teacher: Can we make the pieces more complicated?

(illustration 6-4-16)
(Cut an oddly shaped piece from the right side of a square and put the piece on the left side. Cut an oddly shaped piece from the bottom of the same square and put the piece on the top. Then show the shape tessellating.)

Teacher: Will a rectangle tessellate with pieces cut and moved?

(illustration 6-4-17)
(Cut an oddly shaped piece from the right side of a rectangle and put the piece on the left side. Cut an oddly shaped piece from the bottom of the same rectangle and put the piece on the top. Then show the shape tessellating.)

Teacher: Will a parallelogram?

(illustration 6-4-18)
(Cut an oddly shaped piece from the right side of a parallelogram and put the piece on the left side. Cut an oddly shaped piece from the bottom of the same parallelogram and put the piece on the top. Then show the shape tessellating.)

Teacher: Will a hexagon?

(illustration 6-4-19)
(Cut an oddly shaped piece from three adjacent sides of a hexagon and put the pieces on the opposite and parallel sides. Then show the shape tessellating.)

Teacher: Will a cut-up scalene quadrilateral tessellate as well?

(illustration 6-4-20)
(Use the scalene quadrilateral from the "rotation" illustration earlier in this lesson to demonstrate the effect of cutting pieces out of a rotated tessellation. Cut an oddly shaped piece from the right side of a quadrilateral and put the piece on the left side.)

Teacher: Can the piece we cut stay on the side we cut it from?

(illustration 6-4-21)
(Demonstrate the sequence of cutting a piece from one side of a square and adding it to the same side. Put a mark half way on the side of a square. Cut a piece from the lower half. Attach the piece to the upper half. Show how the piece tessellates by flipping or rotating the side onto itself. Second example: Equilateral triangle with a piece cut from half of one side pasted on the same side. (An example can be found on page 93 of Teaching Tessellating Art.) Show the triangle tessellating. Third example: Equilateral triangle with pieces cut from half of one side pasted on the same side. Repeat for all three sides of the triangle. Show the triangle tessellating.)

What other questions might we ask?

Teacher: What happens if we say our tessellations do not have to be regular? What happens if we use two different polygons and not just one?

(illustration 6-4-22)
(Octagon, square "semi-regular tessellation". Hexagon, equilateral triangle, "semi-regular tessellation".)

Teacher: What else can you find out?

Geometry is visual explorations. The questions are simple. The learning is complex.

Procedures...

As our students create new shapes with which to tessellate, we teach them procedures that smooth the process of creativity.

Teacher: When you cut a piece from your shape, the easiest way to make sure the cut-out part fits correctly on another side is to start cutting the piece at one corner of the side and cut all the way to the other corner.

(illustration 6-4-23)
(Illustration of what is meant by corner-to-corner cutting and placing.)

Teacher: Another way to make sure that the piece that you cut from one side is placed correctly on another is to use grid or graph paper for your shape. The lines or grids on the piece you cut can help you place the piece in the correct position on the other side.

(illustration 6-4-24)
(Illustration of what is meant by using grid or graph paper to place cut-outs on another side.)

Teacher: If you color one side of your shape before you cut, you can tell by the different colors for each side if you accidentally flip the piece you cut to its back side. Pieces cut and moved have to stay right side up.

(illustration 6-4-25)
(Illustrate a cut-out piece colored differently on one side moved correctly and flipped incorrectly after being moved.)

Teacher: Once you cut and move a piece, you can tape the piece in its new position to hold it where you want it as you trace. Do your taping carefully, so the piece taped on does not stick out farther that the hole into which it is to fit.

(illustration 6-4-26)
(Demonstrate carefully and not carefully taped pieces.)

What students tessellate and trace, they color in imaginatively.

(illustration 6-4-27)
(Examples of student tessellations from previous illustrations colored in imaginatively. Show a range from basic A-B, A-B coloring of squares and rectangles through Escher-like drawing of repetitive characters for elaborate tessellations of imaginatively created new shapes.)

Why tessellate?...

Why look for symmetry in designs? Why tessellate? Why is this mathematics?

Throughout the ages, humans have sought to comprehend the order of the universe. Mathematics is the language that we use to describe the patterns in the order that we find. Symmetry is the name we give to the regularity and balance we discover in nature's shapes. Symmetry is in the molecular structure of crystals, in the design of flakes of snow, in the peddles of a flower blossom, in the face of any animal we see.

Tessellations teach our students that mathematics is more than numbers. Mathematics is visual patterns as well. Geometric tessellations teach us to see patterns visually.

How many of us know what fractals are? Order in chaos cannot always be quantified, but it can be visualized. Fractals are the unique patterns of the seemingly unpredictable movements, or chaos, of nature. Fractals are a new geometry. Do we know how fractals are used in mathematics? Do we know uses that fractals have in life?

(illustration 6-4-28)
(Fractal design)

> A fractal is a colored image that mathematically models how well things survive in their environments. Life forms survive through a process of constant change, change that must take place within reasonable bounds if the organism is not to be destroyed. The environments that an organism lives in, such as a forest or atmosphere, play a critical role in maintaining this healthy state of change. Those environments which are the most conducive to long term survival are colored [one color] in the fractal image. Those which are not conductive to life, such as an atmosphere filled with smog or a forest filled with acid rain, are colored [differently] according to how long the organism is able to live in them before it can no longer survive.
>
> Science uses fractals to model and predict the survival of everything from hurricanes to intergalactic nebulae. In fact anything whose survival is dependent on its surroundings can be modeled with this new [geometry]. For example, the hurricane exists within the rest of the atmosphere of Earth. If the remaining atmosphere, where there is no hurricane, were to be removed, the hurricane would quickly disappear. Thus the survival of a hurricane is very much dependent on the bright and calm sunny days on the other side of the planet. Changing the atmosphere around the hurricane changes the life and course of the hurricane itself. In this way, science uses fractal mathematics to better understand and cope with the [seeming chaos] of the wonderful natural world that surrounds us.
>
> From: *What is a Fractal?,* Dynamical Systems Laboratory Films, 1990.

What in our educational backgrounds has prepared us for using or even understanding fractals? What was in the math we learned besides the numbers? Tessellations prepare our students for a world of visual mathematics in a future we cannot anticipate.

In *Mathematics... a Way of Thinking* a chapter is devoted to logical thinking. (*Mathematics... a Way of Thinking*, Chapter 20, Tangrams - Logical Thinking.) The logical thinking chapter is a chapter on geometry. The basis for the chapter's lessons are semi-regular tessellations. Students fill specific planes with polygons with no gaps or overlapping, or explain why they cannot. Logical thinking is an integral part of Mathematics. Tessellations are an integral part of thinking logically.

We admire the mathematical beauty of nature. We admire the mathematical beauty of designs that humans make. We can recreate this beauty in our class. This beauty is the beauty of geometry. As we teach our students tessellations, they learn to associate the beauty of nature and the beauty of design with the beauty that is math.

Lesson Five

Purpose	Learn that math and art are not separate subjects.
Summary	We teach art as we always do. In Patterns & Connections, Lesson Five, we pointed out the patterns to be seen. We now point to the geometric connections to be made, as well.
Materials	Assorted art supplies.
Topic	Extending the tessellations from Lesson Four.
Topic	Name symmetry.
Topic	Snow flakes.
Topic	3-D straw constructions.
Topic	What math can we see in other art that is a part of the art we teach?
Homework	Children who have art materials at home can continue their art work there.

Art awareness...

What is art?

The use of skill and imagination in the production of things of beauty.

Webster's New Collegiate Dictionary

As we dance, there is symmetry in the glides and steps of our motions. We use numbers to count out the steps. Music has form and symmetry as well. Mathematics is in the notes that we use to record the songs we sing and the music we play. Mathematics is in all the arts. We emphasize the link between mathematics and any form of art at every opportunity. Our students find symmetry and create tessellations in geometry. We carry their geometric experiences into art. Where does the mathematics lesson end and the art lesson begin? Endings and beginnings are dividing lines for learning only found in school.

Ask a class of kindergarten children "Who can sing?" or, "Who can dance?" or, "Who can draw?" and every child is apt to answer, "I can!" Children come to us believing they are artists, everyone.

The arts are areas of instruction that permit us to keep everyone's belief in himself or herself strong. In the arts, learning can be given all the time it needs. There are no timed tests, no facts to memorize, no answers to mark right or wrong. There is only inventiveness and creativity. Art encourages differences—freedom of expression is allowed. Each child's efforts and accomplishments have value independently of any other child's.

Renaissance Masters used perspective geometry to give depth to paintings. Beethoven and Mozart symphonies are written in whole, half, quarter and eighth notes. Tempo is a rate of speed. Dance is a patterned set of steps set to music. We need only make ourselves aware of the math that is already present in the art. We need only to make ourselves aware, as well, of the art that is present in the math.

Below are four examples of how we may carry our students' geometric experiences with symmetry and tessellation into art. As our students create, we point out the mathematics that we see. As we learn to look, what we see will multiply.

Tessellating art...

As our students create tessellations in geometry, we expand their opportunities to tessellate in art.

(illustration 6-5-1)
(Examples of tessellations on wrapping paper, on posters and on clothes. Include specific instructions on how to put tessellating designs on clothing.)

Name symmetry...

(illustration 6-5-2)

(Include step by step visual instructions for creating name symmetry designs. First step: Fold drawing paper in half the long way. Second step. Write or print name (first or first and last, depending on ability level) in heavy crayon on the fold line of the drawing paper. Third step: Refold paper along original line and rub the crayon name onto the opposite side of the drawing paper until the image of the name written in crayon appears in mirror image (lightly) on the opposite side. Hard object like scissor handle or piece of wood make good rubbing devices. Fourth step: Draw the light mirror image of the name in darkly with crayon. Fifth step: Color in the symmetrical totem pole like name images imaginatively.)

Flakes of snow...

(illustration 6-5-3)
(Classic "snow flake" art lesson. Illustrate the process step by step. Fold paper in six folds. Cut notches into the folded paper. Unfold paper to see the snow flake created.)

Straws in 3-D...

(illustration 6-5-4)
(Demonstrate the making of a 3-D triangular straw kite, complete with tissue paper. Include in the caption that this is an example of a three dimensional tessellation.)

Awareness summary...

The art examples above are lessons we may have seen before. The art we teach does not have to change to link the art to math. The mathematics is already in the arts we teach. The change is in our awareness of the connections to be made.

Lesson Six

Purpose	Learn to draw three-dimensional shapes.
Summary	Students learn to draw three-dimensional shapes so that what is drawn identifies the shape.
Materials	Geoblocks or other 3-D shapes, drawing paper.
Topic	Draw one block. Others identify the block drawn.
Topic	Draw two blocks. Others identify the blocks drawn.
Topic	Draw three and more blocks. Others identify the blocks drawn.

Visual imagery...

(illustration 6-6-1)
(Drawings of a top view, front view and side view of a Geoblock. The three views are on the overhead. The views are labeled top, front, side, as appropriate.)

Teacher: Here are three different views of a Geoblock. Show me the block you think these drawings represent.

(illustration 6-6-2)
(Geoblock placed next to the drawing of a Geoblock.)

Teacher: Is the block you have shown me the only Geoblock my drawing can represent?

Geometry is visual imagery. We use geometry to describe in pictures what we see.

Teacher: Draw a picture of a Geoblock so that someone else in class can use your picture to identify the Geoblock you drew.
Student: Can we trace around the block on our paper if we want?
Teacher: You may trace if you wish.
Student: Do we have to use three different views?
Teacher: Anyway you draw the block is fine as long as someone else can find your block and not a different one.
Once you finish with your drawing, put your name on your paper and put your block inside your desk. When everyone has finished, I'll come around and collect the drawings that you've done.

(illustration 6-6-3)